Inverted Cassini Ovals

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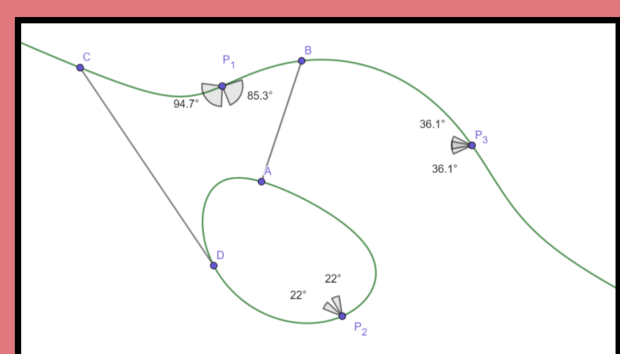
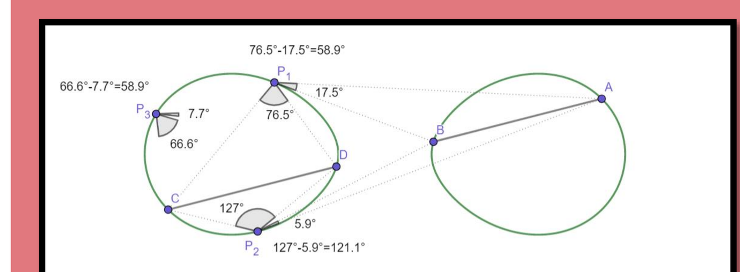
Introduction

Where can one person stand so that my poster takes up half of their vision? In two dimensions this question is answered by the inscribed angle theorem. The person would have to stand on a circle whose diameter is my poster.

Now, suppose I had two posters that are not next to one another. Where can someone stand so my posters takes up half of their vision? Moreover, where can someone stand so they see my first poster more than the other or vice versa? These questions are the focus of this research.

An Apollonian cubic (as shown on the right) is part of the locus of points which see two segments at equal or supplementary angles[3].





A Cassini Oval is a collection of points for which the products of two distances from given foci to these points is constant

It has been shown that two chords that form a diameter of a Cassini oval subtend angles whose difference is constant at any point of the oval (See above figure)[1].

The angle differences, sums, and equalities above can be generalized to oriented angle sums whose values are between $-\pi$ and π .

Define the Oriented Dissected θ -Isoptic (θ -Disoptic) to be the points which view two closed curves at oriented angles which sum to θ or $\theta - \pi$ for some $\theta \in (0, \pi]$.

From the above it follows that the Apollonian cubic is an oriented π -disoptic for two segments and a Cassini oval is an oriented θ -disoptic for congruent collinear segments.

Primary Question:

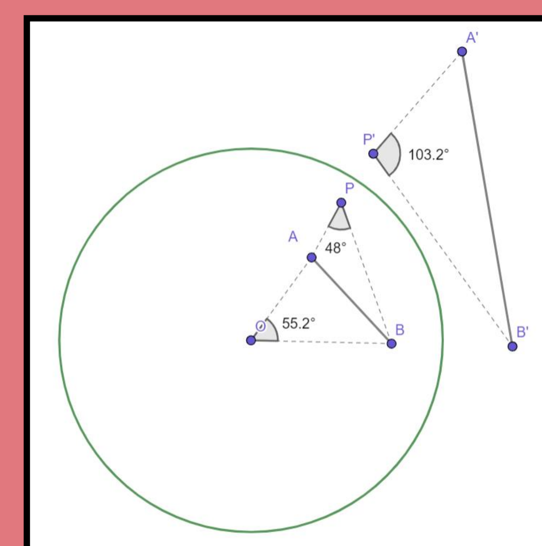
For two given segments what curve describes the Oriented θ-Disoptic?

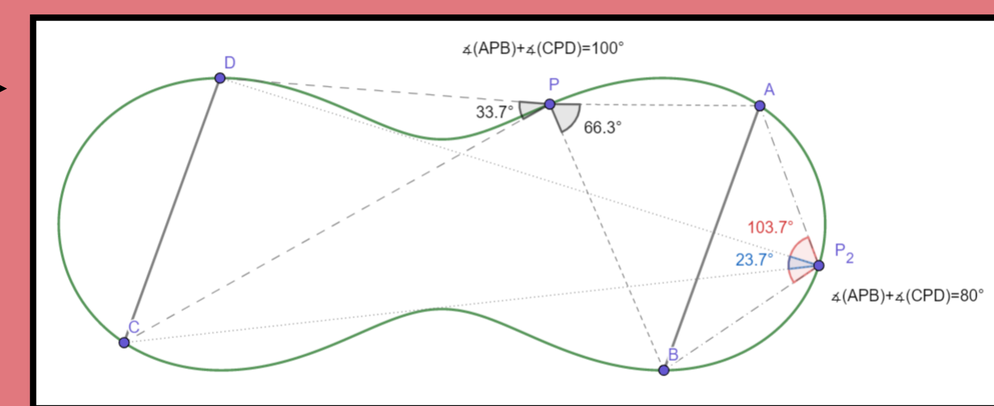
Background and Notation

-Let $\measuredangle(APB) = \measuredangle APB$ if the path from A to P to B is clockwise.
-Many of the results on the curves are derived from the principal value of a complex number and the Mobius Transform.
-The circle inversion of a point P over some circle with center O and radius r is the point P' on $\overrightarrow{OP}$ such that the product of OP and OP' is r^2.

Methodology

Theorem: Given segments AB and CD such that ABCD is a parallelogram the θ -disoptic of AB and CD is a Cassini oval, a right hyperbola, or two perpendicular lines.





Lemma: For unique points A,P, B, and O and their respective inversions A',P', and B' over some circle with center O we have

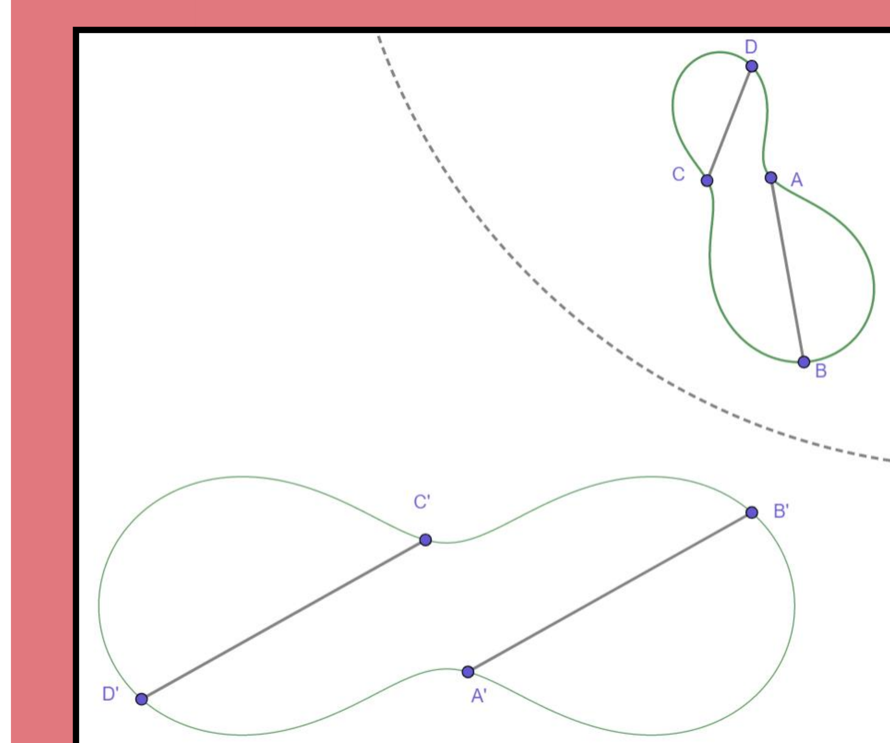
$$\measuredangle(B'P'A') = \measuredangle(BOA) + \measuredangle(APB) \bmod(-\pi, \pi]$$

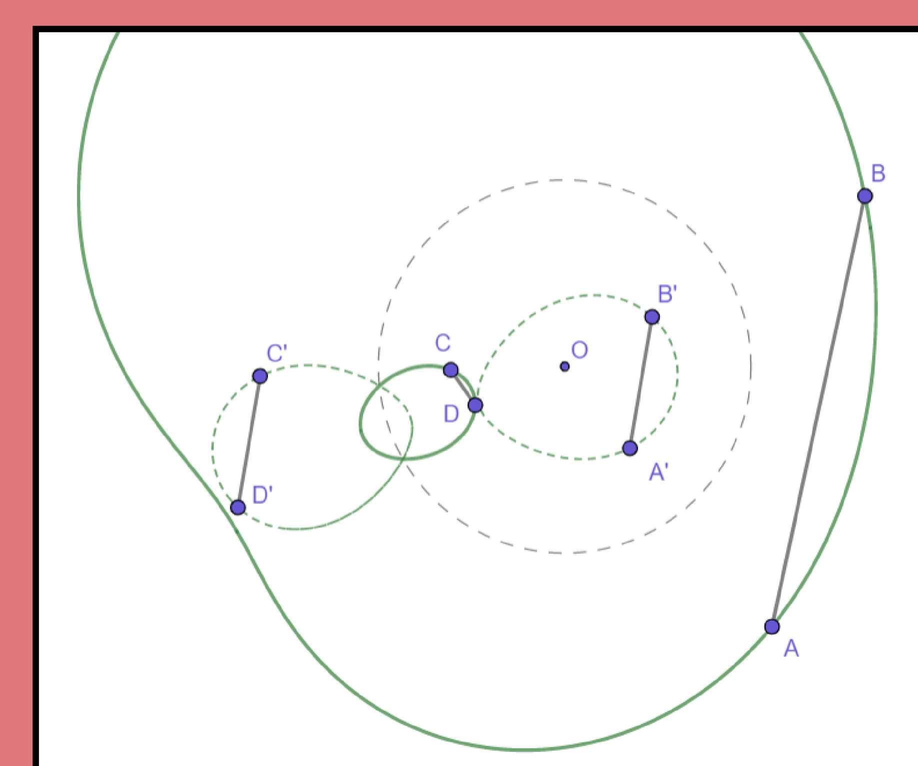
Lemma: For any four points A,B, C, and D there is some circle which inverts the points, so they form a parallelogram ABCD, labeled in that orientation.

Major Result

By applying the above Lemmas and Theorem we arrive at this major conclusion

Theorem: The oriented θ -disoptic of two segments is the circle inversion of a Cassini oval or two perpendicular lines.





In simpler terms, if a person had half of their vision taken up by my two posters, they would be standing on the arc of an inverted Cassini Oval.

Discussion and Further Research

- We arrived at a classification of oriented θ –disoptics for two segments.
  - For three segments, in general there is no circle inversion analog for the second lemma. So a classification for n segments will have more interesting curves.
  - Other research exists on equioptics[2] (similar to π – disoptics) of two closed curves. Our methods here become cumbersome when considering any closed curves. This is because the points of tangency are moving.
- Apollonian cubics have a natural group structure using the chord and tangent addition rule. We are interested to see if there is some similar group structure on inverted Cassini ovals.
- In essence this research extends the inscribed angle theorem to two segments.
  - In this vein, the angle for the oriented disoptic can be determined via the foci and center of inversion of the inverted Cassini oval.

Citations

[1] Mathematical Questions and Solutions in Continuation of the Mathematical Columns of "the Educational Times", Vol. 51, pg 41, F. Hodgson, 1889, Digitized Nov. 18 2009

[2] B. Odenhal, Equioptic Curves of Conic Sections, *Journal for Geometry and Graphics*, Vol 14 (2010), No. 1, 29-43.

[3] P. Pamfilos and A. Thoma, Apollonain Cubics: An Application of Group Theory to a Problem in Euclidean Geometry, *Mathematics Magezine*, Vol. 72, No. 5 (Dec. 1999), pp. 356-366, Taylor&Francis, https://www.jstor.org/stable/2690791

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