Understanding Radar from a Mathematical Standpoint

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May 2022

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Introduction

Radar systems are fundamentally complex systems which are gaining relevance as technology advances.

Radar systems give way to many applications, including but not limited to:

- Vehicle traffic, aircraft in the air, ships at sea (Surveillance)
- Vehicle or weapon guidance systems (Navigation)
- Picking up signatures through all types of weather (Signatures)

In this project we analyze the way radar systems operate to adequately measure their capabilities to other technological systems.

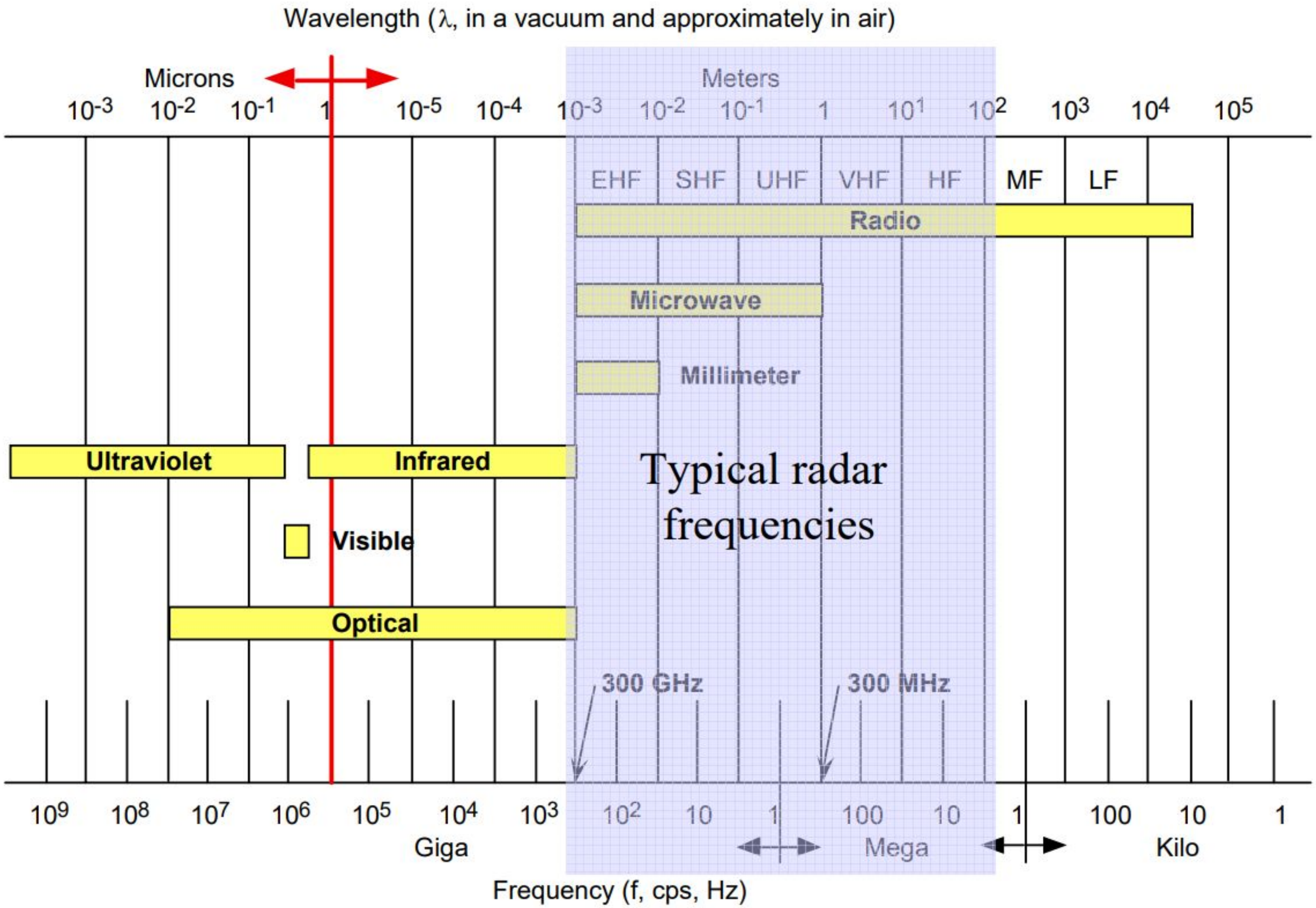
Radar Detection and Ranging

Radar, or Radar Detection and Ranging, is a detection system which uses electromagnetic waves to gather information about objects.

Normal radar functions include the measurements of:

- Range (from pulse delay),
- Velocity of an object (Doppler frequency shifts).

By increasing the complexity of a radar, we may extend the functionality of the system to include signature analysis and inverse scattering to which we may measure object size, shape, and even material composition.

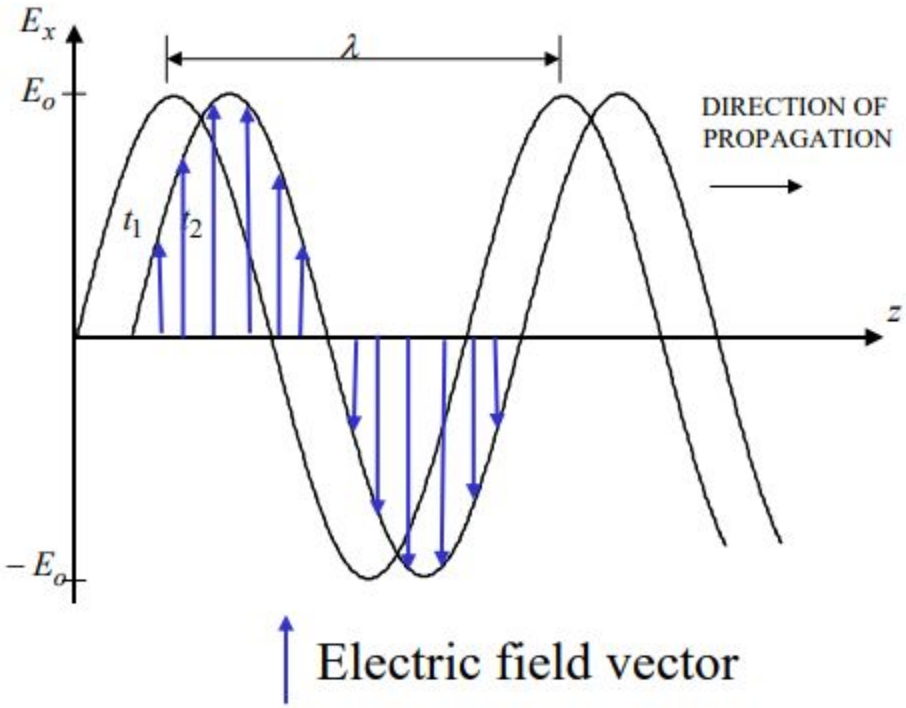


Wavelength (λ, in a vacuum and approximately in air)
Microns
Meters
10-3 10-2 10-1 1 10-5 10-4 10-3 10-2 10-1 1 101 102 103 104 105
EHF SHF UHF VHF HF MF LF
Radio
Microwave
Millimeter
Ultraviolet
Infrared
Visible
Optical
Typical radar frequencies
300 GHz
300 MHz
109 108 107 106 105 104 103 102 10 1 100 10 1 100 10 1
Giga
Mega
Kilo
Frequency (f, cps, Hz)

Properties of Waves

As you can see, the propagation of our waves can be represented mathematically as a sine function with properties:

- Wavelength of λ
- Amplitude of E_o
- Polarized in the x-direction
- $t_1 \rightarrow t_2$ is an example of phase shift



A Simple Radar

A burst of oscillating electromagnetic energy generated from a waveform generator propagates through free space through the transmitter of an antenna. The pulse of energy scatters (reradiates) off of any objects it encounters. The scattered energy returns to the radar receiver. The range R to a detected object can then be calculated by

$$R = \frac{ct}{2}$$

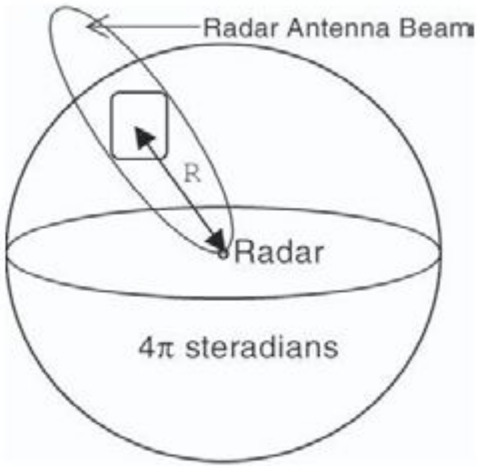
where c is the speed of light (3×10^6 m/s) and t is the time.

The Radar Range Equation

Now that the principles of radar have been established, we can now derive the radar range equation. Here we have an isotropic simple radar with electromagnetic pulse of peak power *P* radiating into free space.

Focusing the pulse, we introduce the gain of the transmit antenna *G*, and the energy density received at range *R*, giving us the expression

$$\frac{PG}{4\pi R^2}$$



The amount of energy a target intercepts and radiates back to the antenna is given by the radar cross section, σ. So the energy reradiated from the target is given by

$$\frac{PG\sigma}{4\pi R^2}$$

Assuming that the location of the transmitted signal and the receive antenna of area A are the same, the energy which we can assign as signal S, gives us

$$S = \frac{PGA\sigma}{(4\pi)^2 R^4}$$

Because of noise, the energy received by the system is adversely affected. We represent this using the product of the Boltzmann constant *K* and the temperature *T*. Adding on losses throughout the whole system *L*, the radar range equation can be rewritten as

$$\frac{S}{N} = \frac{PGA\sigma}{(4\pi)^2 R^4 KTL}$$

Where *S*/*N* represents a singular pulse of the signal-to-noise ratio.

Since radars do not typically operate off of one simple signal-to-noise ratio pulse, we can change the power *P* in the numerator to reflect the average power P_{ave} multiplied by the time *t* over all pulses. Thus,

$$\frac{S}{N} = \frac{P_{ave} t G A \sigma}{(4\pi)^2 R^4 K T L}$$

Since the gain *G* is directly related to the aperture of the antenna *A* and λ is the radar signal wavelength, we can substitute *A* for

$$A = \frac{G_r \lambda^2}{4\pi}$$

Giving us,

$$\frac{S}{N} = \frac{P_{ave} t G_t G_r \lambda^2 \sigma}{(4\pi)^3 R^4 KTL}$$

where G_r is the gain of the receive aperture and G_t is the gain of the transmit antenna. If the receive filter does not match to the transmit waveform, we need to account for the receiver filter bandwidth B. Hence, the radar range equation,

$$\frac{S}{N} = \frac{P G_t G_r \lambda^2 \sigma}{(4\pi)^3 R^4 KTBL}$$

Note that the radar range equation does not only have one form. Rewriting the equation can give us the maximum range of a radar as seen below.

$$R_{max} = \left[\frac{P G_t G_r \lambda^2 \sigma}{(4\pi)^3 \left(\frac{S}{N}\right) KTBL} \right]^{\frac{1}{4}}$$

Now that we have derived the radar range equation, we can now solve for each of these variables in the equation for any radar system.

Question:

If a radar with transmit peak power P = 1 MW peak power and antenna gain G = 1000 (30 dB) irradiates a target with an RCS σ = 1 m^2 target at a range R = 500 km range, what power density arrives back at the radar antenna?

P = 1 MW = 0.001 W, G = 1000, RCS (σ) = 1 m^2, R = 500 km = 500000 m

Given the parameters, we use the simplified form of the radar range equation.

$$\frac{PG\sigma}{4\pi R^2}$$

So, plugging everything in,

$$\frac{PG\sigma}{4\pi R^2} = \frac{(0.001)(1000)(1)}{4\pi(500000)^2} = 3.183 \times 10^{-13}$$

we get that the power density arriving back at the system is equal to 3.183×10^{-13} W/m^2.

Radar Cross Section (RCS)

Radar cross section is defined by the measure of electromagnetic energy (in dBsm) which scatters off an object back to the receiver.

Although the RCS of an object can be solved using Maxwell's equations, solving for various boundary conditions of complex objects can pose a challenge.

So, RCS is obtained by measuring the object's cross section and comparing it to reference objects.

*Computer programs can break down complex cross sections into multiple simple surfaces preserving the properties of the electromagnetic wave to calculate RCS.

But for referencing objects, calculating RCS becomes rather simple.

Assuming that the target we are measuring the RCS of does not reradiate the waves isotropically. So we assign it gain, as

$$\sigma = GA$$

where A is given by the area of the object as seen by the radar. Remembering that gain is given by the equation

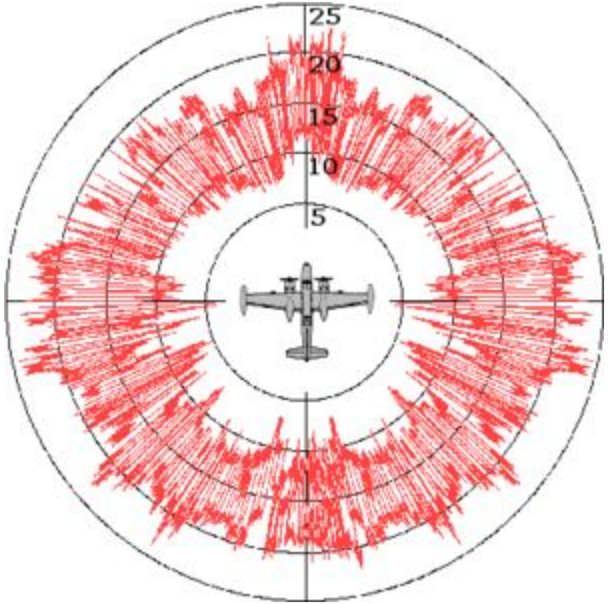
$$G = \frac{4\pi A}{\lambda^2} \quad \lambda < \sqrt{A}$$

We end up with the RCS of an object is given by

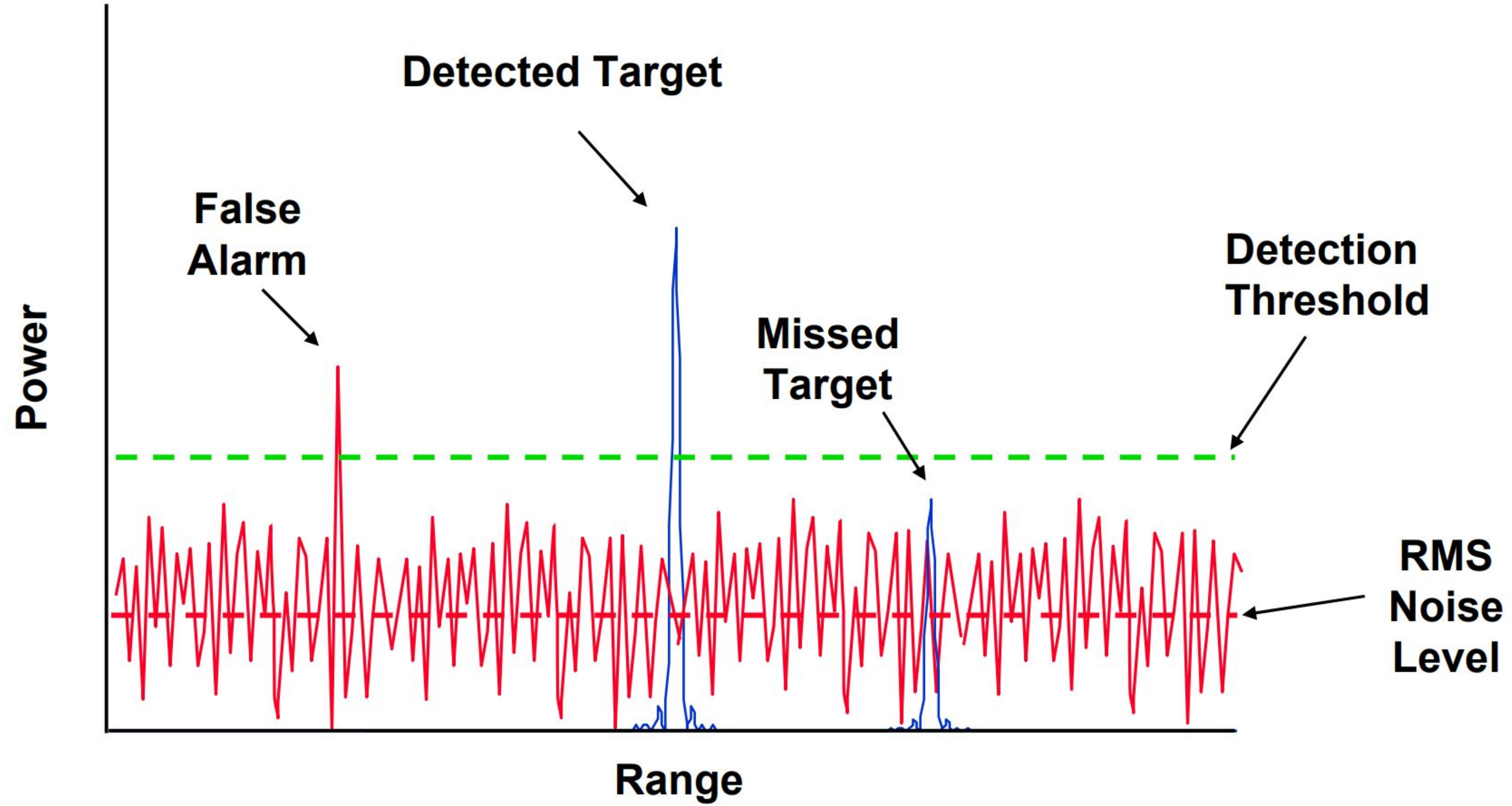
$$\sigma = \frac{4\pi A^2}{\lambda^2}$$

Shape	Aspect Angle	RCS	Constraints	Symbols
Sphere	Any	πr^2	$\frac{2\pi r}{\lambda} > 10$	r = radius
	Any	$8\pi^3\left(\frac{V^2}{\lambda^4}\right)$	$\frac{2\pi r}{\lambda} < 1$	V = volume
Flat plate	Broadside	$\frac{4\pi A^2}{\lambda^2}$	$\frac{2\pi a}{\lambda} > 1$	A = area a = dimension
Cone	Axial	$\frac{\lambda^2}{16\pi}\tan^4(\theta)$		θ = cone half angle
Cone sphere	Nose-on	$0.1\lambda^2$	$\frac{2\pi r_{nose\ tip}}{\lambda} < 1$	$r_{nose\ tip}$ = nose tip radius
		$\approx \pi r_{nose\ tip}^2$	$\frac{2\pi r_{nose\ tip}}{\lambda} > 1$	
Truncated cone	Nose-on	$\approx \pi r_{flat\ back}^2 \pm \sigma_{nose\ tip}$	$\frac{2\pi r_{flat\ back}}{\lambda} > 1$	$r_{flat\ back}$ = flat back radius
Cylinder	Broadside	$\frac{\pi d L^2}{\lambda}$	$\frac{2\pi d}{\lambda} > 1$	d = diameter L = length
Long wire	Broadside	$\pi L^2\left(\frac{d}{2\lambda}\right)^{0.57}$	$\frac{L}{\lambda} >> 1$	d = diameter L = length
Resonant dipole	Broadside	$\frac{\pi}{4}\lambda^2$	$L = \frac{\lambda}{2}$	L = length
	Spherically random	$0.15\lambda^2$		
Cloud of resonant dipoles	Random	$0.15\ N\lambda^2$	$L = \frac{\lambda}{2}$	N = number of dipoles
Convex surfaces	Random	$\pi r_x r_y$	$(\lambda < \ < r_x)$ and r_y	r_x = radius of curvature x-plane r_y = radius of curvature, y-plane
Dihedral corner reflector	Axis of	$\frac{8\pi a^4}{\lambda^2}$	$\lambda < a$	a = dimension
Trihedral corner reflector	Axis of	$\frac{12\pi a^4}{\lambda^2}$	$\lambda < a$	a = dimension
Triangular corner reflector	Axis of	$\frac{4\pi a^4}{3\lambda^2}$		a = dimension

Although the equation we have just derived applies to only flat cross sections, other cross sections are derived in a similar fashion.



Signal-to-Noise Ratio (SNR)



What is Doppler Shift?

Doppler Shift

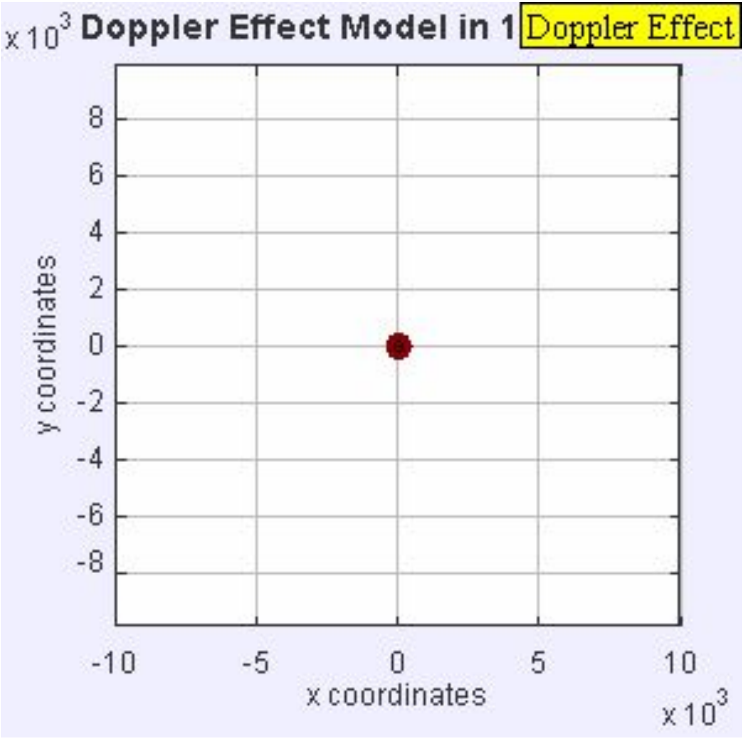
Doppler shift is the change in frequency of a wave related to an "observer."

This phenomenon occurs when each consecutive wave is emitted from a position closer or further away from the observer. Hence, the change in frequency.

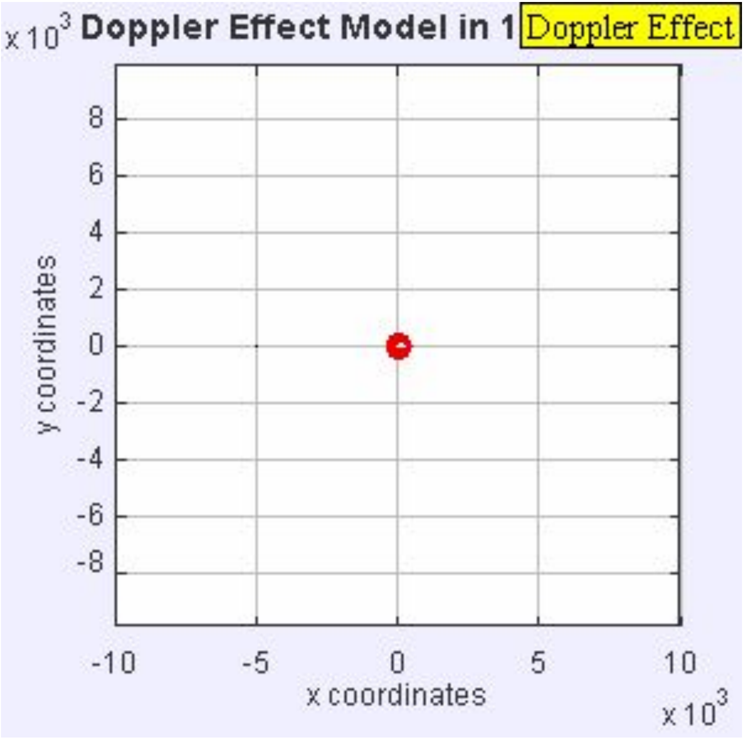
Example) Vehicle honking the horn drives past.

The observed frequency and the change in frequency can be given by the two equations, respectively.

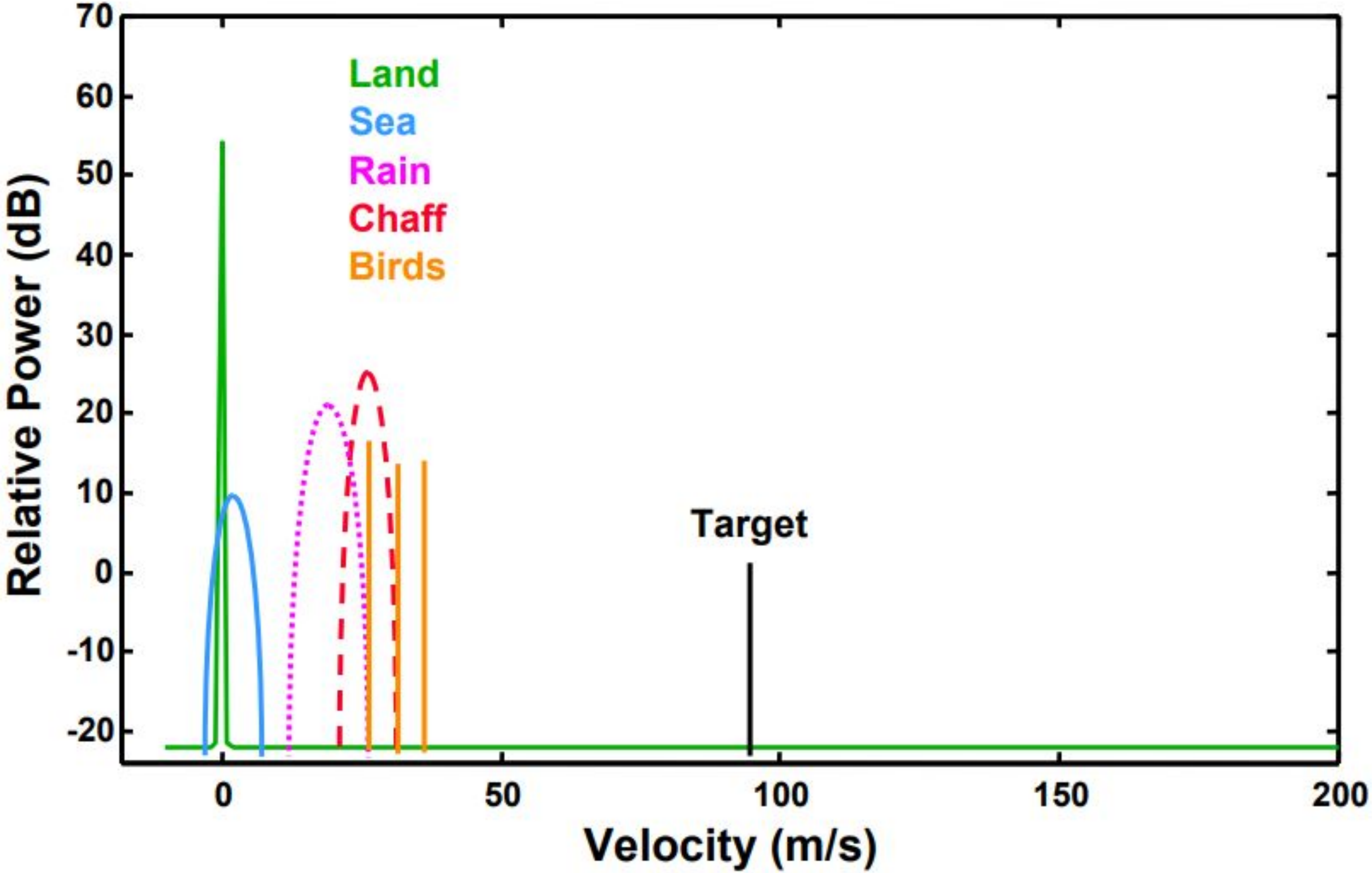
$$f = \left(1 + \frac{\Delta v}{c}\right) f_0 \qquad \Delta f = \frac{\Delta v}{c} f_0$$



x 10^3 Doppler Effect Model in 1
Doppler Effect
8
6
4
2
0
-2
-4
-6
-8
y coordinates
-10
-5
0
5
10
x coordinates
x 10^3



x 10^3 Doppler Effect Model in 1
Doppler Effect
8
6
4
2
0
-2
-4
-6
-8
y coordinates
-10
-5
0
5
10
x coordinates
x 10^3



Land
Sea
Rain
Chaff
Birds
Target
Relative Power (dB)
Velocity (m/s)
70
60
50
40
30
20
10
0
-10
-20
0
50
100
150
200

Thank you for your time!

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